Critical points and classification

Given the following function:

$$f(x, y) = 4y^3 + 12y^2x + 4x^3 - 12x$$

Determine the critical points and classify them as maxima, minima, or saddle points.

Solution

Calculate the partial derivatives:

- $f_x = \dfrac{\partial f}{\partial x} = 12y^2 + 12x^2 - 12$
- $f_y = \dfrac{\partial f}{\partial y} = 12y^2 + 24xy$

Set the partial derivatives equal to zero:

$$\begin{cases} 12y^2 + 12x^2 - 12 = 0 & (1) \\ 12y^2 + 24xy = 0 & (2) \end{cases}$$

Simplify:

- From (1): $y^2 + x^2 - 1 = 0 \implies x^2 + y^2 = 1$
- From (2): $y(y + 2x) = 0$

Solving the system of equations:

Case 1: $y = 0$
Substitute into the circle equation:

$$x^2 + 0^2 = 1 \implies x^2 = 1 \implies x = \pm 1$$

Critical points: $(1, 0)$ **and** $(-1, 0)$

Case 2: $y + 2x = 0 \implies y = -2x$
Substitute into the circle equation:

$$x^2 + (-2x)^2 = 1 \implies x^2 + 4x^2 = 1 \implies 5x^2 = 1 \implies x = \pm \frac{1}{\sqrt{5}}$$

Calculate y:

$$y = -2x = -2\left(\pm \frac{1}{\sqrt{5}}\right) = \pm \frac{2}{\sqrt{5}}$$

Critical points:

- $\left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)$
- $\left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$

Classification of the critical points
Calculate the second-order partial derivatives:

- $f_{xx} = \frac{\partial^2 f}{\partial x^2} = 24x$
- $f_{yy} = \frac{\partial^2 f}{\partial y^2} = 24y + 24x$
- $f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = 24y$

Hessian determinant: $D = f_{xx} f_{yy} - (f_{xy})^2$

Critical point $(1, 0)$**:**

- $x = 1$, $y = 0$
- $f_{xx} = 24(1) = 24$
- $f_{yy} = 24(0) + 24(1) = 24$
- $f_{xy} = 24(0) = 0$
- $D = (24)(24) - (0)^2 = 576$

Since $D > 0$ and $f_{xx} > 0$, there is a **local minimum** at $(1, 0)$.

Critical point $(-1, 0)$**:**

- $x = -1$, $y = 0$
- $f_{xx} = 24(-1) = -24$
- $f_{yy} = 24(0) + 24(-1) = -24$
- $f_{xy} = 24(0) = 0$
- $D = (-24)(-24) - (0)^2 = 576$

Since $D > 0$ and $f_{xx} < 0$, there is a **local maximum** at $(-1, 0)$.

Critical point $\left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)$**:**

- $x = \frac{1}{\sqrt{5}}$, $y = -\frac{2}{\sqrt{5}}$
- $f_{xx} = 24\left(\frac{1}{\sqrt{5}}\right) = \frac{24}{\sqrt{5}}$
- $f_{yy} = 24\left(-\frac{2}{\sqrt{5}}\right) + 24\left(\frac{1}{\sqrt{5}}\right) = -\frac{48}{\sqrt{5}} + \frac{24}{\sqrt{5}} = -\frac{24}{\sqrt{5}}$
- $f_{xy} = 24\left(-\frac{2}{\sqrt{5}}\right) = -\frac{48}{\sqrt{5}}$
- $D = \left(\frac{24}{\sqrt{5}}\right)\left(-\frac{24}{\sqrt{5}}\right) - \left(-\frac{48}{\sqrt{5}}\right)^2 = -\frac{576}{5} - \frac{2304}{5} = -\frac{2880}{5} = -576$

Since $D < 0$, there is a **saddle point** at $\left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)$.

Critical point $\left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$**:**

- $x = -\frac{1}{\sqrt{5}}$, $y = \frac{2}{\sqrt{5}}$
- $f_{xx} = 24\left(-\frac{1}{\sqrt{5}}\right) = -\frac{24}{\sqrt{5}}$
- $f_{yy} = 24\left(\frac{2}{\sqrt{5}}\right) + 24\left(-\frac{1}{\sqrt{5}}\right) = \frac{48}{\sqrt{5}} - \frac{24}{\sqrt{5}} = \frac{24}{\sqrt{5}}$
- $f_{xy} = 24\left(\frac{2}{\sqrt{5}}\right) = \frac{48}{\sqrt{5}}$

- $D = \left(-\frac{24}{\sqrt{5}}\right)\left(\frac{24}{\sqrt{5}}\right) - \left(\frac{48}{\sqrt{5}}\right)^2 = -\frac{576}{5} - \frac{2304}{5} = -\frac{2880}{5} = -576$

  Since $D < 0$, there is a **saddle point** at $\left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$.

  The critical points and their classification are:

- $(1, 0)$: **Local minimum**.
- $(-1, 0)$: **Local maximum**.
- $\left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)$: **Saddle point**.
- $\left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$: **Saddle point**.